

Complex Hadamard matrices attached to even orthogonal schemes of class 4

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December 6, 2016

Abstract

A complex Hadamard matrix is a square matrix W with complex entries of absolute value 1 satisfying $WW^* = nI$, where $*$ stands for the Hermitian transpose and I is the identity matrix of order n . In this paper, we give constructions of complex Hadamard matrices in the Bose–Mesner algebra of a certain 4-class symmetric association scheme. Moreover, we determine the Nomura algebras to show that the resulting matrices are not decomposable into nontrivial generalized tensor products.

1 Introduction

A complex Hadamard matrix is a square matrix W with complex entries of absolute value 1 satisfying $WW^* = nI$, where $*$ stands for the Hermitian transpose and I is the identity matrix of order n . They are the natural generalization of real Hadamard matrices. Complex Hadamard matrices appear frequently in various branches of mathematics and quantum physics.

A type-II matrix, or an inverse orthogonal matrix, is a square matrix W with nonzero complex entries satisfying $WW^{(-)\top} = nI$, where $W^{(-)}$ denotes the entrywise inverse of W . Obviously, a complex Hadamard matrix is a type-II matrix.

In [7], we gave a method to find a complex Hadamard matrix in the Bose–Mesner algebra of a symmetric association scheme. Applying this result, we classified complex Hadamard matrices in the Bose–Mesner algebra of a certain 3-class association scheme. In this paper, we construct certain complex Hadamard matrices in the Bose–Mesner algebra of a 4-class association scheme $(X, \{R_i\}_{i=0}^4)$ with the first eigenmatrix:

$$P = \begin{bmatrix} 1 & \frac{1}{2}(q-2)q^{2m-1} & \frac{1}{2}q^{2m} & q(q^{2m-2}-1) & q-2 \\ 1 & \frac{1}{2}(q-2)q^{m-1} & \frac{1}{2}q^m & -(q-1)(q^{m-1}+1) & q-2 \\ 1 & -\frac{1}{2}(q-2)q^{m-1} & -\frac{1}{2}q^m & -(q-1)(q^{m-1}-1) & q-2 \\ 1 & \frac{1}{2}q^m & -\frac{1}{2}q^m & 0 & -1 \\ 1 & -\frac{1}{2}q^m & \frac{1}{2}q^m & 0 & -1 \end{bmatrix}, \quad (1)$$

where q and m are positive integers with $q \geq 4$ and $m \geq 2$. Then $|X| = q^{2m} - 1$, R_4 is a disconnected relation, and R_2 defines a strongly regular graph. If q is a power of 2, an even orthogonal scheme is an example of an association scheme with the first eigenmatrix (1) (see [3, Chapter 12.1]). If $m = 1$, then $R_3 = \emptyset$, and this scheme reduces to an even orthogonal scheme of class 3 which we considered in [7].

For a type-II matrix $W \in M_X(\mathbb{C})$ and $a, b \in X$, we define column vectors Y_{ab} by setting

$$(Y_{ab})_x = \frac{W_{xa}}{W_{xb}} \quad (x \in X).$$

The *Nomura algebra* $N(W)$ of W is the algebra of matrices in $M_n(\mathbb{C})$ such that Y_{ab} is an eigenvector for all $a, b \in X$. It is shown in [9, Theorem 1] that the Nomura algebra is a Bose–Mesner algebra.

Throughout this paper, we denote by $\mathfrak{X} = (X, \{R_i\}_{i=0}^4)$ a symmetric association scheme with the first eigenmatrix (1). Let A_0, A_1, A_2, A_3, A_4 be the adjacency matrices of \mathfrak{X} . Let $w_0 = 1, w_1, w_2, w_3, w_4$ be nonzero complex numbers, and set

$$W = \sum_{j=0}^4 w_j A_j, \quad (2)$$

$$a_{i,j} = \frac{w_i}{w_j} + \frac{w_j}{w_i} \quad (0 \leq i < j \leq 4). \quad (3)$$

The main purpose of this paper is to prove the following:

Theorem 1. *Assume that*

$$w_4 = 1. \quad (4)$$

- (i) *Assume $w_1 = 1$. Then, the matrix W in (2) is a complex Hadamard matrix if and only if*

$$w_2^2 + \frac{2(q^{2m} - 2)}{q^{2m}} w_2 + 1 = 0 \quad \text{and} \quad w_3 = 1.$$

- (ii) *Assume*

$$a_{0,1} = \frac{2(q^{4m-2} - (q+2)q^{2m-1} + 2)}{(q^{2m-1} + q - 2)q^{2m-1}}. \quad (5)$$

Then, the matrix W in (2) is a complex Hadamard matrix if and only if

$$w_2 = -\frac{(q-1)q^{2m-1}w_1 + q^{2m-1} + q - 2}{(q^{2m-1} - 1)q}, \quad (6)$$

$$w_3 = 1.$$

Theorem 2. *Let W be a complex Hadamard matrix given in (i) and (ii) of Theorem 1. The algebra $N(W)$ coincides with the linear span of I and J . In particular, W is not equivalent to a nontrivial generalized tensor product.*

The reason for the assumption (4) is as follows: Calculating the conditions under which the matrix (2) becomes a complex Hadamard matrix experimentally for small q and m , we find that (4) is fulfilled, or

$$(iii) \quad a_{0,4} = 2(q^{2m} - 6)/(q^{2m} - 4), \text{ or}$$

$$(iv) \quad a_{0,4} \text{ is a zero of a polynomial of degree 9.}$$

For the case (iii) with $m = 2, 3$, we have $w_1 = w_3 = w_4$. Therefore, this case reduces to the case in which the matrix W given in (2) belongs to the Bose–Mesner algebra of the strongly regular graph defined by R_2 . However, it seems to be difficult to prove $w_1 = w_3 = w_4$ for arbitrary $m \geq 4$.

For the case (iv), we verified that the polynomial in $a_{0,4}$ of degree 9 is an irreducible polynomial for $m = 2, \dots, 9$ and $q = 2^s$ with $2 \leq s \leq 10000$. However, it seems difficult to determine the polynomial of degree 9 satisfied by $a_{0,4}$ in general. For example, for $(q, m) = (4, 2)$, if the matrix (2) is a complex Hadamard matrix, then $a_{0,4}$ is a zero of the polynomial

$$\begin{aligned} p(x) = & x^9 - \frac{235721}{1785}x^8 - \frac{17957726593}{62475}x^7 + \frac{33219815829811}{937125}x^6 \\ & - \frac{12554318926285933}{4685625}x^5 + \frac{29740292638491103}{312375}x^4 \\ & - \frac{696525696876795217}{187425}x^3 + \frac{851886544261448041}{37485}x^2 \\ & - \frac{124583919439776136}{2499}x + \frac{30888835313436500}{833}. \end{aligned}$$

It can be shown that $p(x)$ has only one real root in $(-2, 2)$ by using Sturm's theorem. Then, by using Lemmas 1 and 2 below, there exist w_1, w_2, w_3, w_4 such that (2) is a complex Hadamard matrix.

Under the hypothesis of (4), we find that $a_{0,1} = 2$ or $a_{0,1}$ is given by (5), or

(v) $a_{0,1}$ is a zero of a polynomial of degree 4.

It seems to be difficult to determine w_1, w_2, w_3 for the case (vi) for arbitrary q and m . For example, for $(q, m) = (4, 2)$, if the matrix (2) is a complex Hadamard matrix, then w_1, w_2, w_3 are given by the following:

$$\begin{aligned} w_1^2 + \frac{21s - 7140 \pm 85t}{176}w_1 + 1 &= 0, \\ w_2 &= -\frac{64(w_1^2 - 1)}{127w_1 + 64a_{0,2}}, \\ w_3 &= \frac{90(w_1^2 - 1)}{90a_{1,3}w_1 - 4s + 1117}, \\ a_{0,2} &= \frac{43s - 14620 \pm 85t}{352}, \\ a_{1,3} &= \frac{21s - 1848 \mp (4s + 1253)t}{2640}, \\ s &= \sqrt{104899}, \\ t^2 &= \frac{8s - 2591}{3}. \end{aligned}$$

2 Preliminaries

We define a polynomial in three indeterminates X, Y, Z as follows:

$$g(X, Y, Z) = X^2 + Y^2 + Z^2 - XYZ - 4.$$

We define a polynomial in six indeterminates $X_{0,1}, X_{0,2}, X_{0,3}, X_{1,2}, X_{1,3}, X_{2,3}$ as follows:

$$h(X_{0,1}, X_{0,2}, X_{0,3}, X_{1,2}, X_{1,3}, X_{2,3}) = \det \begin{bmatrix} 2 & X_{0,1} & X_{0,2} \\ X_{0,1} & 2 & X_{1,2} \\ X_{0,3} & X_{1,3} & X_{2,3} \end{bmatrix}.$$

For a finite set N and a positive integer k , we denote by $\binom{N}{k}$ the collection of all k -element subsets of N .

Lemma 1 ([7, Lemma 4]). *Let $N = \{0, 1, \dots, d\}$, $N_3 = \binom{N}{3}$ and $N_4 = \binom{N}{4}$. Let $a_{i,j}$ ($0 \leq i, j \leq d$, $i \neq j$) be complex numbers satisfying*

$$a_{i,j} = a_{j,i} \quad (0 \leq i < j \leq d), \quad (7)$$

$$g(a_{i,j}, a_{j,k}, a_{i,k}) = 0 \quad (\{i, j, k\} \in N_3), \quad (8)$$

$$h(a_{i,j}, a_{i,k}, a_{i,\ell}, a_{j,k}, a_{j,\ell}, a_{k,\ell}) = 0 \quad (\{i, j, k, \ell\} \in N_4). \quad (9)$$

Assume

$$a_{i_0, i_1} \neq \pm 2 \quad \text{for some } i_0, i_1 \text{ with } 0 \leq i_0 < i_1 \leq d. \quad (10)$$

Let w_{i_0}, w_{i_1} be nonzero complex numbers satisfying

$$\frac{w_{i_0}}{w_{i_1}} + \frac{w_{i_1}}{w_{i_0}} = a_{i_0, i_1}. \quad (11)$$

Then for complex numbers w_i ($0 \leq i \leq d$, $i \neq i_0, i_1$), the following are equivalent:

(i) for all i, j with $0 \leq i, j \leq d$ and $i \neq j$,

$$\frac{w_j}{w_i} + \frac{w_i}{w_j} = a_{i,j} \quad (12)$$

(ii) for all i, j with $0 \leq i \leq d$, $i \neq i_0, i_1$,

$$w_i = \frac{w_{i_1}^2 - w_{i_0}^2}{a_{i_1, i} w_{i_1} - a_{i_0, i} w_{i_0}}. \quad (13)$$

Moreover, if one of the two equivalent conditions (i), (ii) is satisfied, $a_{i,j}$ ($0 \leq i < j \leq d$) are all real and

$$-2 < a_{i_0, i_1} < 2, \quad (14)$$

then $|w_i| = |w_j|$ for $0 \leq i < j \leq d$.

We let \mathcal{A} denote a symmetric Bose–Mesner algebra with adjacency matrices A_0, A_1, \dots, A_d . Let n be the size of the matrices A_i , and we denote by

$$P = (P_{i,j})_{\substack{0 \leq i \leq d \\ 0 \leq j \leq d}}$$

the first eigenmatrix of \mathcal{A} . Then the adjacency matrices are expressed as

$$A_j = \sum_{i=0}^d P_{i,j} E_i \quad (j = 0, 1, \dots, d),$$

where $E_0 = \frac{1}{n}J, E_1, \dots, E_d$ are the primitive idempotents of \mathcal{A} .

Let w_0, w_1, \dots, w_d be nonzero complex numbers, and set

$$W = \sum_{j=0}^d w_j A_j \in \mathcal{A}. \quad (15)$$

Lemma 2 ([7, Lemma 7]). Let $X_{i,j}$ ($0 \leq i < j \leq d$) be indeterminates and let e_k be the polynomial defined by

$$e_k = \sum_{0 \leq i < j \leq d} P_{k,i} P_{k,j} X_{i,j} + \sum_{i=0}^d P_{k,i}^2 - n \quad (k = 1, \dots, d). \quad (16)$$

Let $a_{i,j}$ ($0 \leq i, j \leq d$, $i \neq j$) and w_i ($0 \leq i \leq d$) be complex numbers. Assume that $w_i \neq 0$ for all i with $0 \leq i \leq d$ and that (12) holds. Then the following statements are equivalent:

- (i) the matrix W given by (15) is a type-II matrix,
- (ii) $(a_{i,j})_{0 \leq i < j \leq d}$ is a common zero of e_k ($k = 1, \dots, d$).

Moreover, if one of the two equivalent conditions (i), (ii) is satisfied, $a_{i,j} \in \mathbb{R}$ for all i, j with $0 \leq i < j \leq d$, and (14) holds for some i_0, i_1 with $0 \leq i_0 < i_1 \leq d$, then W is a scalar multiple of a complex Hadamard matrix.

We now describe the proof of Theorem 1 briefly. Let A_0, A_1, A_2, A_3, A_4 be the adjacency matrices of an association scheme \mathfrak{X} with the first eigenmatrix (1). Let $w_0 = 1, w_1, w_2, w_3, w_4$ be nonzero complex numbers, and W be the matrix defined by (2). For $i, j \in \{0, 1, 2, 3, 4\}$, define $a_{i,j}$ by (3). We write

$$\mathbf{a} = (a_{0,1}, a_{0,2}, a_{0,3}, a_{0,4}, a_{1,2}, a_{1,3}, a_{1,4}, a_{2,3}, a_{2,4}, a_{3,4}) \quad (17)$$

for brevity. Consider the polynomial ring

$$R = \mathbb{C}[X_{0,1}, X_{0,2}, X_{0,3}, X_{0,4}, X_{1,2}, X_{1,3}, X_{1,4}, X_{2,3}, X_{2,4}, X_{3,4}]. \quad (18)$$

In Section 3, we first assume that W is a complex Hadamard matrix. Then by Lemmas 1 and 2, \mathbf{a} is a common zero of the polynomials

$$g(X_{i,j}, X_{i,k}, X_{j,k}) \quad (\{i, j, k\} \in \binom{\{0, 1, 2, 3, 4\}}{3}), \quad (19)$$

$$h(X_{i,j}, X_{i,k}, X_{i,l}, X_{j,k}, X_{j,l}, X_{k,l}) \quad (\{i, j, k, l\} \in \binom{\{0, 1, 2, 3, 4\}}{4}), \quad (20)$$

$$e_k \quad (k \in \{1, 2, 3, 4\}). \quad (21)$$

Let \mathcal{I} be the ideal of R generated by these polynomials. Calculating the ideal generated by \mathcal{I} and $X_{0,4} - 2$, we find (i) and (ii) of Theorem 1.

Conversely, we assume that $w_4 = 1$ and w_1, w_2, w_3 are given in Theorem 1. Then, to show that the matrix W given in (2) is a complex Hadamard matrix, we check that \mathbf{a} defined by (3), (17) is a zero of the polynomials (19), (20), (21). Moreover, we check that $-2 < a_{i_0, i_1} < 2$ holds for some i_0, i_1 with $0 \leq i_0 < i_1 \leq 4$.

All the computer calculations in this paper were performed with the help of Magma [2]. In order to facilitate computations covering all possible values of the integer q , we perform the computations in the polynomial ring with 12 variables $q, r = q^m$ and $X_{i,j}$ over the field of rational numbers, rather than the ring (18). The results valid for this generic setting are also valid for arbitrary integers q and m .

3 Proof of Theorem 1

Recall $q \geq 4$ and $m \geq 2$, and \mathcal{I} is the ideal of the polynomial ring R generated by the polynomials (19), (20), and (21). For the remainder of this section, we assume that $a_{0,4} = 2$, that is, $w_4 = 1$. Let \mathcal{I}_1 denote the ideal generated by \mathcal{I} and $X_{0,4} - 2$. For Lemmas 3–5 we assume that \mathbf{a} defined in (17) is a common zero of the polynomials in \mathcal{I}_1 .

Lemma 3. *We have*

$$a_{1,2} = -\frac{2(q^{2m} - 2)}{q^{2m}}. \quad (22)$$

Proof. We can verify that \mathcal{I}_1 contains $X_{1,2} + 2(q^{2m} - 2)/q^{2m}$. Hence we have (22). \square

Lemma 4. *Assume $a_{0,1} = 2$. Then, (w_1, w_2, w_3) is given in (i) of Theorem 1.*

Proof. Let \mathcal{I}_2 denote the ideal generated by \mathcal{I}_1 and $X_{0,1} - 2$. Then we can verify that \mathcal{I}_2 contains $(X_{0,3} - 2)^2$, that is, $a_{0,3} = 2$. Hence $w_1 = w_3 = 1$. Since $a_{1,2}$ is given in (22), the matrix W given in (2) belongs to the Bose–Mesner algebra of the strongly regular graph defined by R_2 . From [4] we have the condition of w given in (i) of Theorem 1. \square

Lemma 5. *Assume that $a_{0,1}$ is given in (5). Then, (w_1, w_2, w_3) is given in (ii) of Theorem 1.*

Proof. Let \mathcal{I}_3 denote the ideal generated by \mathcal{I}_1 and $X_{0,1} - a_{0,1}$. Then we can verify that \mathcal{I}_3 contains $q(q^{2m-1} + q - 2)X_{0,2} + 2(q^{2m} - q^2 + 2q - 2)$, that is,

$$a_{0,2} = -\frac{2(q^{2m} - q^2 + 2q - 2)}{q(q^{2m-1} + q - 2)} \quad (23)$$

Let \mathcal{I}_4 denote the ideal generated by \mathcal{I}_3 and $p_1(X_{0,2})$. Then we can verify that \mathcal{I}_4 contains $X_{0,3} - 2$, that is, $w_3 = 1$. From (13) we obtain

$$w_2 = \frac{w_1^2 - 1}{a_{1,2}w_1 - a_{0,2}}.$$

Since $w_1^2 - a_{0,1}w_1 + 1 = 0$, we have (6) from (22), (23). \square

Proof of Theorem 1. Suppose that the matrix (2) is a complex Hadamard matrix. For $i, j \in \{0, 1, 2, 3, 4\}$, define $a_{i,j}$ by (3). Let \mathbf{a} be given in (17). Then \mathbf{a} is a common zero of the polynomials in \mathcal{I}_1 by Lemma 2. From Lemmas 4, 5 we have (i) and (ii) of Theorem 1.

Conversely, assume that w_1, w_2, w_3 , and w_4 are given in Theorem 1. Then, we show that the matrix given in (2) is a complex Hadamard matrix. To do this, we check that \mathbf{a} defined by (3) is a zero of the polynomials (19), (20), and (21), and (w_1, w_2, w_3) are complex numbers of absolute value 1. The latter condition is satisfied if $-2 < a_{i_0, i_1} < 2$ holds for some i_0, i_1 with $0 \leq i_0 < i_1 \leq 4$.

Case (i) is done by [4].

Next consider Case (ii). From (3), (5), and (6) we have (22) and (23). Then we have

$$\mathbf{a} = (a_{0,1}, a_{0,2}, 2, 2, a_{1,2}, a_{0,1}, a_{0,1}, a_{0,2}, a_{0,2}, 2).$$

This is a zero of the polynomials (19), (20), and (21). It is easy to check that $0 < a_{0,1} < 2$. \square

4 Proof of Theorem 2

Since $q^{2m} - 1$ is a composite, there are uncountably many inequivalent complex Hadamard matrices of order $q^{2m} - 1$ by [6]. Indeed, such matrices can be constructed using generalized tensor products [8]. We show that none of our complex Hadamard matrices is equivalent to a nontrivial generalized tensor product. This is done by showing that the Nomura algebra of our complex Hadamard matrices has dimension 2. According to [8], the Nomura algebra of a nontrivial generalized tensor product of type-II matrices is imprimitive, and this is never the case when it has dimension 2.

Recall $q \geq 4$ and $m \geq 2$. The intersection matrices $B_i = (p_{ij}^k)$ ($i = 0, \dots, 4$) of \mathfrak{X} are given by the following:

$$\begin{aligned}
 B_0 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \\
 B_1 &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{(q-2)q^{2m-1}}{2} & \frac{(q-2)^2 q^{2m-2}}{4} & \frac{(q-2)^2 q^{2m-2}}{4} & \frac{(q-2)^2 q^{2m-2}}{4} & \frac{(q-4)q^{2m-1}}{4} \\ 0 & \frac{(q-2)q^{2m-1}}{4} & \frac{(q-2)q^{2m-1}}{4} & \frac{(q-2)q^{2m-1}}{4} & \frac{q^{2m}}{4} \\ 0 & \frac{(q-2)(q^{2m-2}-1)}{4} & \frac{(q-2)(q^{2m-2}-1)}{4} & \frac{(q-2)q^{2m-2}}{2} & 0 \\ 0 & \frac{q-4}{2} & \frac{q-4}{2} & 0 & 0 \end{bmatrix}, \\
 B_2 &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{(q-2)q^{2m-1}}{4} & \frac{(q-2)q^{2m-1}}{4} & \frac{(q-2)q^{2m-1}}{4} & \frac{q^{2m}}{4} \\ \frac{q^{2m}}{2} & \frac{q^{2m}}{4} & \frac{q^{2m}}{4} & \frac{q^{2m}}{4} & \frac{q^{2m}}{4} \\ 0 & \frac{q(q^{2m-2}-1)}{2} & \frac{q(q^{2m-2}-1)}{2} & \frac{q^{2m-1}}{2} & 0 \\ 0 & \frac{q}{2} & \frac{q-2}{2} & 0 & 0 \end{bmatrix}, \\
 B_3 &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{(q-2)(q^{2m-2}-1)}{2} & \frac{(q-2)(q^{2m-2}-1)}{2} & \frac{(q-2)q^{2m-2}}{2} & 0 \\ 0 & \frac{q(q^{2m-2}-1)}{2} & \frac{q(q^{2m-2}-1)}{2} & \frac{q^{2m-1}}{2} & 0 \\ q(q^{2m-2}-1) & q^{2m-2}-1 & q^{2m-2}-1 & q^{2m-2}-2q+1 & q(q^{2m-2}-1) \\ 0 & 0 & 0 & q-2 & 0 \end{bmatrix}, \\
 B_4 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{q-4}{2} & \frac{q-2}{2} & 0 & 0 \\ 0 & \frac{q}{2} & \frac{q-2}{2} & 0 & 0 \\ 0 & 0 & 0 & q-2 & 0 \\ q-2 & 0 & 0 & 0 & q-3 \end{bmatrix}.
 \end{aligned}$$

Lemma 6. *The algebra $N(W)$ is symmetric.*

Proof. Suppose that $N(W)$ is not symmetric. Then by [9, Proposition 6(i)], there exists $(b, c) \in X^2$ with $b \neq c$ such that

$$\sum_{x \in X} \frac{W_{x,b}^2}{W_{x,c}^2} = 0.$$

This is equivalent to

$$\sum_{j,k} p_{jk}^i \frac{w_j^2}{w_k^2} = 0$$

for some $i \in \{1, 2, 3, 4\}$. Using the notation (3), we have

$$\begin{aligned} \sum_{j,k} p_{jk}^i \frac{w_j^2}{w_k^2} &= \sum_{j < k} p_{jk}^i \left(\frac{w_j^2}{w_k^2} + \frac{w_k^2}{w_j^2} \right) + \sum_{j=0}^4 p_{jj}^i \\ &= \sum_{j < k} p_{jk}^i \left(\left(\frac{w_j}{w_k} + \frac{w_k}{w_j} \right)^2 - 2 \right) + \sum_{j=0}^4 p_{jj}^i \\ &= \sum_{j < k} p_{jk}^i (a_{j,k}^2 - 2) + \sum_{j=0}^4 p_{jj}^i. \end{aligned} \tag{24}$$

It can be verified by computer that (24) is nonzero for each of the cases (i)–(ii) in Theorem 1. \square

Since $N(W)$ is symmetric, the adjacency matrices of $N(W)$ are the $(0, 1)$ -matrices representing the connected components of the Jones graph defined as follows (see [9, Sect. 3.3]). The *Jones graph* of a type-II matrix $W \in M_X(\mathbb{C})$ is the graph with vertex set X^2 such that two distinct vertices (a, b) and (c, d) are adjacent whenever $\langle Y_{ab}, Y_{cd} \rangle \neq 0$, where $\langle \cdot, \cdot \rangle$ denotes the ordinary (not Hermitian) scalar product.

Proof of Theorem 2. We claim that (x, y) and (x, z) belong to the same connected component in the Jones graph whenever $(x, y), (y, z), (z, x) \in R_4$. Indeed, if (x, y) and (x, z) belong to different connected components, then (y, x) and (z, x) belong to different connected components by Lemma 6. In particular,

$$\langle Y_{xy}, Y_{xz} \rangle = \langle Y_{yx}, Y_{zx} \rangle = 0.$$

Let

$$c_{i,j,k} = |\{u \in X \mid (x, u) \in R_i, (y, u) \in R_j, (z, u) \in R_k\}|.$$

Since $p_{1,3}^4 = p_{2,3}^4 = 0$, we have

$$c_{1,j,k} + c_{2,j,k} = c_{j,1,k} + c_{j,2,k} = c_{j,k,1} + c_{j,k,2} = p_{j,k}^4 \tag{25}$$

for $j, k \in \{1, 2\}$. Then we have

$$\sum_{i,j,k=0}^4 c_{i,j,k} \frac{w_i^2}{w_j w_k} = \sum_{i,j,k=0}^4 c_{i,j,k} \frac{w_j w_k}{w_i^2} = 0. \tag{26}$$

Since the rank of the coefficient matrix in (25) is 7, we have one degree of freedom in (25). Combining (25) and (26), it can be verified by computer that these conditions give rise to a polynomial equation in q which has no solution in positive integers $q \geq 4$ for each of the cases (i) and (ii) in Theorem 1.

Therefore, we have proved the claim. This, together with Lemma 6, implies that, for each equivalence class C of the equivalence relation $R_0 \cup R_4$, $(C \times C) \cap R_4$ belongs to the same connected component in the Jones graph.

Let C and C' be two distinct equivalence classes of $R_0 \cup R_4$. We claim that, for any $(x, z) \in C \times C'$, there exist $y \in C$ such that $\langle Y_{xy}, Y_{xz} \rangle \neq 0$, and there exist $y' \in C'$ such that $\langle Y_{y'z}, Y_{xz} \rangle \neq 0$.

Suppose $(x, z) \in R_1$ and $\langle Y_{xy}, Y_{xz} \rangle = 0$ for all $y \in R_4(x)$. Then

$$\begin{aligned}
0 &= \sum_{y \in R_4(x)} \langle Y_{xy}, Y_{xz} \rangle \\
&= \sum_{y \in R_4(x)} \sum_{u \in X} (Y_{xy})_u (Y_{xz})_u \\
&= \sum_{y \in R_4(x)} \sum_{u \in X} \frac{W_{xu}^2}{W_{yu} W_{zu}} \\
&= \sum_{y \in R_4(x)} \sum_{i,j=0}^4 \sum_{u \in R_i(x) \cap R_j(z)} \frac{W_{xu}^2}{W_{yu} W_{zu}} \\
&= \sum_{i,j=0}^4 \sum_{u \in R_i(x) \cap R_j(z)} \sum_{k=0}^4 \sum_{y \in R_4(x) \cap R_k(u)} \frac{w_i^2}{w_k w_j} \\
&= \sum_{i,j=0}^4 \sum_{u \in R_i(x) \cap R_j(z)} \sum_{k=0}^4 p_{4k}^i \frac{w_i^2}{w_k w_j} \\
&= \sum_{i,j,k=0}^4 p_{ij}^1 p_{4k}^i \frac{w_i^2}{w_k w_j}.
\end{aligned}$$

It can be verified by computer that this leads to a polynomial equation in q which has no solution in positive integers $q \geq 4$. Set $\ell \in \{2, 3\}$. Similarly, suppose $(x, z) \in R_\ell$ and $\langle Y_{xy}, Y_{xz} \rangle = 0$ for all $y \in C$. Then

$$\sum_{i,j,k=0}^4 p_{ij}^\ell p_{4k}^i \frac{w_i^2}{w_k w_j} = 0,$$

and again this leads to a contradiction. Thus, there exists $y \in C$ such that $\langle Y_{xy}, Y_{xz} \rangle \neq 0$. Switching the role of x and z , we see that there exists $y' \in C'$ such that $\langle Y_{y'z}, Y_{xz} \rangle \neq 0$. Therefore, we have proved the claim.

Since C and C' are arbitrary, the claim shows that, in the Jones graph, R_4 is contained in a single connected component, and that every element $(x, z) \in R_1 \cup R_2 \cup R_3$ is adjacent to an element of R_4 . Thus, $R_1 \cup R_2 \cup R_3 \cup R_4$ is a connected component of the Jones graph. Therefore, $\dim N(W) = 2$. \square

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A Verification by Magma

Proof of Theorem 1

```
d:=4;
d2s:=&cat[[[i,j]:j in [i+1..d]]:i in [0..d-1]];
d2:=[Seqset(s):s in d2s];
R:=PolynomialRing(Rationals(),#d2+3);
X:=func<i,j|R.Position(d2,{i,j})>;
q:=R.(#d2+1);
r:=R.(#d2+2);
nz1:=R.(#d2+3);
NZ1:=nz1*(q-1)-1;
qm:=q*r;

g:=func<i,j,k|X(i,j)^2+X(i,k)^2+X(j,k)^2-X(i,j)*X(i,k)*X(j,k)-4>;
h:=func<i,j,k,l|(X(k,l)^2-4)*X(i,j)
  -X(k,l)*(X(k,i)*X(l,j)+X(k,j)*X(l,i))
  +2*(X(k,i)*X(k,j)+X(l,i)*X(l,j))>;

eigenP:=Matrix(R,5,5,[
1,1/2*qm*r*(q-2),1/2*qm^2,q*(r^2-1),q-2,
1,1/2*r*(q-2),1/2*qm,-(r+1)*(q-1),q-2,
1,-1/2*r*(q-2),-1/2*qm,(r-1)*(q-1),q-2,
1,1/2*qm,-1/2*qm,0,-1,
1,-1/2*qm,1/2*qm,0,-1
]);
P:=func<i,j|eigenP[i+1,j+1]>;
n:=&+[P(0,i):i in [0..d]];
n eq qm^2-1;

e:=func<i|-n+&+[P(i,j)^2:j in [0..d]]
  +&+[P(i,j[1])*P(i,j[2])*X(j[1],j[2]):j in d2s]>; //eq:21
s3:=[Setseq(x):x in Subsets({0..d},3)];
eq7:=[g(i[1],i[2],i[3]):i in s3] cat
[h(0^i,1^i,2^i,3^i):i in Sym({0..d})] cat
[e(i):i in [1..d]];
I:=ideal<R|eq7>;
```

Proof of Lemma 3

```
I1:=ideal<R|I,X(0,4)-2>;
pa12:=qm^2*X(1,2)+2*(qm^2-2);
pa12 in I1; //Lemma 3
```

Proof of Lemma 4

```
I2:=ideal<R|I1,X(0,1)-2>;
```

```
(r^2-1)^2*(X(0,3)-2)^2 in I2;
```

Proof of Lemma 5

```
pa01:=qm*r*(qm*r+q-2)*X(0,1)-2*(qm^2*r^2-qm^2-2*qm*r+2);
I3:=ideal<R|I1,pa01>;
pa02:=q*(qm*r+q-2)*X(0,2)+2*qm^2-2*q^2+4*q-4;
ff:=(qm^2-1)*(148*r^4-8103*r^2+8214*q^2-46102*q+42957)
*((q+1)*(2*q-1)*qm^4*r^6+(2*q+1)*(q^2-7*q+4)*qm^3*r^5
+(5*q^3+27*q^2-22*q-4)*qm^2*r^4-(q^3+11*q^2+46*q-56)*qm^2*r^2
+8*(q+6)*(q-1)*q*r^2-16*q+16);
pa02*ff in I3;

I4:=ideal<R|I3,pa02>;
(r^2-1)*(X(0,3)-2)^2 in I4;
//Total time: 88023.889 seconds, Total memory usage: 253.50MB
```

Proof of Theorem 1

```
Pqr:=PolynomialRing(Rationals(),2);
Fqr<q,r>:=FieldOfFractions(Pqr);
Rqr<z1>:=PolynomialRing(Fqr);
qm:=q*r;
a01:=2*(qm^2*r^2-(q+2)*qm*r+2)/(qm*r*(qm*r+q-2));
a02:=-2*(qm^2-q^2+2*q-2)/(q*(qm*r+q-2));
a12:=-2*(qm^2-2)/(qm^2);
F<w1>:=FieldOfFractions(Rqr/ideal<Rqr|z1^2-a01*z1+1>);
w1^2-a01*w1+1 eq 0;
w2:=((q-1)*qm*r*w1+qm*r+q-2)/((qm*r-1)*q);
w1/w2+w2/w1 eq a12;
w2+1/w2 eq a02;

d:=4;
d2s:=&cat[[[i,j]:j in [i+1..d]]:i in [0..d-1]];
d2:=[Seqset(s):s in d2s];

R:=PolynomialRing(F,#d2);
X:=func<i,j|R.Position(d2,{i,j})>;
g:=func<i,j,k|X(i,j)^2+X(i,k)^2+X(j,k)^2-X(i,j)*X(i,k)*X(j,k)-4>;
h:=func<i,j,k,l|(X(k,l)^2-4)*X(i,j)
-X(k,l)*(X(k,i)*X(l,j)+X(k,j)*X(l,i))
+2*(X(k,i)*X(k,j)+X(l,i)*X(l,j))>;

eigenP:=Matrix(F,5,5,[
1,1/2*qm*r*(q-2),1/2*qm^2,q*(r^2-1),q-2,
1,1/2*r*(q-2),1/2*qm,-(r+1)*(q-1),q-2,
1,-1/2*r*(q-2),-1/2*qm,(r-1)*(q-1),q-2,
1,1/2*qm,-1/2*qm,0,-1,
```

```

1,-1/2*qm,1/2*qm,0,-1
]);
P:=func<i,j|eigenP[i+1,j+1]>;
n:=%+[P(0,i):i in [0..d]];
n eq qm^2-1;

e:=func<i|-n+%+[P(i,j)^2:j in [0..d]]
  +%+[P(i,j[1])*P(i,j[2])*X(j[1],j[2]):j in d2s]>;
s3:=[Setseq(x):x in Subsets({0..d},3)];
eq7:=[g(i[1],i[2],i[3]):i in s3] cat
  [h(0^i,1^i,2^i,3^i):i in Sym({0..d})] cat
  [e(i):i in [1..d]];

subs1:=[a01,a02,2,2,a12,a01,a01,a02,a02,2];
&and[Evaluate(f,subs1) eq 0:f in eq7];
//Total time: 0.440 seconds, Total memory usage: 32.09MB

```

Proof of Theorem 2

Calculation of the intersection matrices $\{B_i\}_{i=0}^4$:

```

P<c111,c112,c121,c122,c211,c212,c221,c222,w1,w2,q,r>
:=PolynomialRing(Rationals(),12);
F:=FieldOfFractions(P);
qm:=q*r;
n:=qm^2-1;

eigenP:=Matrix(F,5,5,[
1,1/2*qm*r*(q-2),1/2*qm^2,q*(r^2-1),q-2,
1,1/2*r*(q-2),1/2*qm,-(r+1)*(q-1),q-2,
1,-1/2*r*(q-2),-1/2*qm,(r-1)*(q-1),q-2,
1,1/2*qm,-1/2*qm,0,-1,
1,-1/2*qm,1/2*qm,0,-1
]);

intersectionMatrices:=function(P)
d1:=Nrows(P);
n:=%+Eltseq(P[1]);
Q:=n*P^(-1);
return [ Matrix(Parent(Q[1][1]),d1,d1,
  [ [ 1/(n*P[1,k])*%+[ Q[1,1]*P[1,i]*P[1,j]*P[1,k] : l in [1..d1] ]
    : k in [1..d1] ] : j in [1..d1] ]
  ) : i in [1..d1] ];
end function;

B1:=Matrix(F,5,5,[0,1,0,0,0,
qm*r*(q-2)/2,r^2*(q-2)^2/4,r^2*(q-2)^2/4,r^2*(q-2)^2/4,(q-4)*qm*r/4,
0,(q-2)*qm*r/4,(q-2)*qm*r/4,(q-2)*qm*r/4,qm^2/4,

```

```
0, (q-2)*(r^2-1)/2, (q-2)*(r^2-1)/2, (q-2)*r^2/2, 0,
0, 1/2*q-2, 1/2*q-1, 0, 0]);
```

```
B2:=Matrix(F,5,5,[0,0,1,0,0,
0, (q-2)*qm*r/4, (q-2)*qm*r/4, (q-2)*qm*r/4, qm^2/4,
qm^2/2, qm^2/4, qm^2/4, qm^2/4, qm^2/4,
0, q*(r^2-1)/2, q*(r^2-1)/2, 1/2*qm*r, 0,
0, 1/2*q, 1/2*q-1, 0, 0]);
```

```
B3:=Matrix(F,5,5,[0,0,0,1,0,
0, (q-2)*(r^2-1)/2, (q-2)*(r^2-1)/2, (q-2)*r^2/2, 0,
0, q*(r^2-1)/2, q*(r^2-1)/2, 1/2*qm*r, 0,
q*(r^2-1), r^2-1, r^2-1, r^2-2*q+1, q*(r^2-1),
0, 0, 0, q-2, 0]);
```

```
B4:=Matrix(F,5,5,[0,0,0,0,1,
0, 1/2*q-2, 1/2*q-1, 0, 0,
0, 1/2*q, 1/2*q-1, 0, 0,
0, 0, 0, q-2, 0,
q-2, 0, 0, 0, q-3]);
```

```
BB:=[ScalarMatrix(5,F!1),B1,B2,B3,B4];
BB eq intersectionMatrices(eigenP);
pijk:=func<i,j,k|P!BB[i+1][j+1,k+1]>;
```

Proof of Lemma 6

```
isSymNbas:=function(ajk)
  aijs:=[[ajk[1],ajk[2],ajk[3],ajk[4]],
    [1,ajk[5],ajk[6],ajk[7]],
    [1,1,ajk[8],ajk[9]],
    [1,1,1,ajk[10]]];
  aij:=func<i,j|aijs[i+1,j]>;
  ff:=[F|&+[pijk(j,k,i)*(aij(j,k)^2-2):j,k in [0..4]|j lt k]
    +&+[pijk(j,j,i):j in [0..4]]:i in [1..4]];
  return [Numerator(ff[i]):i in [1..4]];
end function;
```

```
x02:=- (2*q^2*r^2-4)/(q^2*r^2);
aa:=[2,x02,2,2,x02,2,2,x02,x02,2];
isSymNbas(aa) eq [ (qm^2-1)*(qm^2-4) :i in [1..4]];
```

```
a01:=2*(qm^2*r^2-(q+2)*qm*r+2)/(qm*r*(qm*r+q-2));
a02:=-2*(qm^2-q^2+2*q-2)/(q*(qm*r+q-2));
a12:=-2*(qm^2-2)/(qm^2);
aa:=[a01,a02,2,2,a12,a01,a01,a02,a02,2];
pp:=qm^5*r+2*(q^2-10*q+14)*qm^3*r+
  q*(q-2)*(q^3-2*q^2+8*q+16)*r^2-4*(q-2)*(q^2-2*q+4);
```

```
isSymNbas(aa) eq [ (qm^2-1)*pp : i in [1..3] ] cat
[ (qm^2-1)*(qm^2-4) ];
```

Proof of Theorem 2

The first claim:

```
varname:=[i,j,k]:i,j,k in [1,2]];
c:=func<i,j,k|R.Position(varname,[i,j,k])>;
w0:=1;

cijk:=function(i,j,k)
  if 0 in {i,j,k} then
    if [i,j,k] in {[0,4,4],[4,0,4],[4,4,0]} then
      return 1;
    else
      return 0;
    end if;
  else
    if 3 in {i,j,k} then
      if {3} eq {i,j,k} then
        return pijk(3,3,4);
      else
        return 0;
      end if;
    else
      if 4 in {i,j,k} then
        if {4} eq {i,j,k} then
          return pijk(4,4,4)-1;
        else
          return 0;
        end if;
      else
        return c(i,j,k);
      end if;
    end if;
  end if;
end function;

fx:=[cijk(1,j,k)+cijk(2,j,k)-pijk(j,k,4):j,k in [1,2]];
fy:=[cijk(j,1,k)+cijk(j,2,k)-pijk(j,k,4):j,k in [1,2]];
fz:=[cijk(j,k,1)+cijk(j,k,2)-pijk(j,k,4):j,k in [1,2]];
fxyz:=fx cat fy cat fz;

a02N1:=(2*q^2*r^2-4);
a02D1:=q^2*r^2;
a021:=a02N1/a02D1;
fa21:=a02D1*w2^2-a02N1*w2+1;
```

```

ww1:=[w0,1,w2,1,1];

alpha1:=func<i|ww1[i+1]>;
ff1:=&+[cijk(i,j,k)*alpha1(i)^2/(alpha1(j)*alpha1(k))
:i,j,k in [0..4]];
gg1:=&+[cijk(i,j,k)*alpha1(j)*alpha1(k)/alpha1(i)^2
:i,j,k in [0..4]];
I1:=ideal<R|[fa21,Numerator(ff1),Numerator(gg1)] cat fxyz>;
(qm^2-1)*(5*qm^6-90*qm^4+313*qm^2-128) in I1;
IsIrreducible(5*qm^6-90*qm^4+313*qm^2-128);

a01N2:=2*(qm^2*r^2-(q+2)*qm*r+2);
a01D2:=(qm*r+q-2)*qm*r;
a01:=a01N2/a01D2;
a022:=-2*(qm^2-q^2+2*q-2)/(q*(qm*r+q-2));
a12:=-2*(qm^2-2)/(qm^2);
fa1:=a01D2*w1^2-a01N2*w1+a01D2;
w2:=(w1^2-1)/(a12*w1-a02);
ww2:=[w0,w1,w2,1,1];

alpha2:=func<i|ww2[i+1]>;
ff2:=&+[cijk(i,j,k)*alpha2(i)^2/(alpha2(j)*alpha2(k))
:i,j,k in [0..4]];
gg2:=&+[cijk(i,j,k)*alpha2(j)*alpha2(k)/alpha2(i)^2
:i,j,k in [0..4]];
I2:=ideal<R|[fa1,Numerator(ff2),Numerator(gg2)] cat fxyz>;
Basis(EliminationIdeal(I2,{q,r}))
eq [qm^10*(qm^2-1)^3*(qm*r+q-2)^4*(qm*r-1)^5];

```

The second claim:

```

t1:=function(l)
return &+[pijk(i,j,l)*pijk(4,k,i)*alpha1(i)^2/(alpha1(j)*alpha1(k))
:i,j,k in [0..4]];
end function;
&and[ (q-2)*(qm^2-1)*(5*qm^6-90*qm^4+313*qm^2-128) in
ideal<R|[fa21,Numerator(t1(l))]> : l in [1..3] ];

s1:=function(l)
return &+[pijk(i,j,l)*pijk(4,k,i)*alpha2(i)^2/(alpha2(j)*alpha2(k))
:i,j,k in [0..4]];
end function;
&and[ qm^7*r*(q-2)*(qm^2-1)^3*(qm*r-1)^5*(qm*r+q-2)^5 in
ideal<R|[fa1,Numerator(s1(l))]> : l in [1..3] ];
//Total time: 34.890 seconds, Total memory usage: 82.78MB

```